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Complex Geometry of Nuclei and Atoms

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Abstract

We propose a new geometrical model of matter, in which neutral atoms are modelled by compact, complex algebraic surfaces. Proton and neutron numbers are determined by a surface's Chern numbers. Equivalently, they are determined by combinations of the Hodge numbers, or the Betti numbers. Geometrical constraints on algebraic surfaces allow just a finite range of neutron numbers for a given proton number. This range encompasses the known isotopes.

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1 Introduction

It is an attractive idea to interpret matter geometrically, and to identify conserved attributes of matter with topological properties of the geometry. Kelvin made the pioneering suggestion to model atoms as knotted vortices in an ideal fluid [1]. Each atom type would correspond to a distinct knot, and the conservation of atoms in physical and chemical processes (as understood in the 19th century) would follow from the inability of knots to change their topology. Kelvin's model has not survived because atoms are now known to be structured and divisible, with a nucleus formed of protons and neutrons bound together, surrounded by electrons. At high energies, these constituents can be separated. It requires of order 1 eV to remove an electron from an atom, but a few MeV to remove a proton or neutron from a nucleus.

Atomic and nuclear physics has progressed, mainly by treating protons, neutrons and electrons as point particles, interacting through electromagnetic and strong nuclear forces [2]. Quantum mechanics is an essential ingredient, and leads to a discrete spectrum of energy levels, both for the electrons and nuclear particles. The nucleons (protons and neutrons) are themselves built from three pointlike quarks, but little understanding of nuclear structure and binding has so far emerged from quantum chromodynamics (QCD), the theory of quarks. These point particle models are conceptually not very satisfactory, because a point is clearly an unphysical idealisation, a singularity of matter and charge density. An infinite charge density causes difficulties both in classical electrodynamics [3] and in quantum field theories of the electron. Smoother structures carrying the discrete information of proton, neutron and electron number would be preferable.

In this paper, we propose a geometrical model of neutral atoms where both the proton number P and neutron number N are topological and none of the constituent particles are pointlike. In a neutral atom the electron number is also P, because the electron's electric charge is exactly the opposite of the proton's charge. For given P, atoms (or their nuclei) with different N are known as different isotopes.

A more recent idea than Kelvin's is that of Skyrme, who proposed a nonlinear field theory of bosonic pion fields in 3+1 dimensions with a single topological invariant, which Skyrme identified with baryon number [4]. Baryon number (also called atomic mass number) is the sum of the proton and neutron numbers, $B = P+N$. Skyrme's baryons are solitons in the field theory, so they are smooth, topologically stable field configurations. Skyrme's model was designed to model atomic nuclei, but electrons can be added to produce a model of a complete atom. Protons and neutrons can be distinguished in the Skyrme model, but only after the internal rotational degrees of freedom are quantised [5]. This leads to a quantised "isospin", with the proton having isospin up ($I_3 = \frac{1}{2}$) and the neutron having isospin down ($I_3 = -\frac{1}{2}$), where I_3 the third component of isospin. The

model is consistent with the well-known Gell-Mann–Nishijima relation [6]

$$Q = \frac{1}{2}B + I_3 \tag{1.1}$$

where Q is the electric charge of a nucleus (in units of the proton charge) and B is the baryon number. Q is integral, because I_3 is integer-valued (half-integer-valued) when B is even (odd). Q equals the proton number P of the nucleus and also the electron number of a neutral atom. The neutron number is $N = \frac{1}{2}B - I_3$. The Skyrme model has had considerable success providing models for nuclei [7, 8, 9, 10]. Despite the pion fields being bosonic, the quantised Skyrmions have half-integer spin if B is odd [11]. But a feature of the model is that proton number and neutron number are not separately topological, and electrons have to be added on.

The Skyrme model has a relation to 4-dimensional fields that provides some motivation for the ideas discussed in this paper. A Skyrmion can be well approximated by a projection of a 4-dimensional Yang–Mills field. More precisely, one can take an SU(2) Yang–Mills instanton and calculate its holonomy along all lines in the (euclidean) time direction [12]. The result is a Skyrme field in 3-dimensional space, whose baryon number B equals the instanton number.

So a quasi-geometrical structure in 4-dimensional space (a Yang–Mills instanton in flat $\mathbb{R}^4$) can be closely related to nuclear structure, but still there is just one topological charge. A next step, first described in the paper [13], was to propose an identification of smooth, curved 4-manifolds with the fundamental particles in atoms – the proton, neutron and electron. Suitable examples of manifolds were suggested. These manifolds were not all compact, and the particles they modelled were not all electrically neutral. One of the more compelling examples was Taub-NUT space as a model for the electron. By studying the Dirac operator on the Taub-NUT background, it was shown how the spin of the electron can arise in this context [14]. There has also been an investigation of multi-electron systems modelled by multi-Taub-NUT space [15, 16]. However, there are some technical difficulties with the models of the proton and neutron, and no way has yet been found to geometrically combine protons and neutrons into more complicated nuclei surrounded by electrons. Nor is it clear in this context what exactly should be the topological invariants representing proton and neutron number.

A variant of these ideas is a model for the simplest atom, the neutral hydrogen atom, with one proton and one electron. This appears to be well modelled by CP^2, the complex projective plane[3]. The fundamental topological property of CP^2 is that it has a generating 2-cycle with self-intersection 1. The second Betti number is $b_2 = 1$, which splits into $b_2^+ = 1$ and $b_2^- = 0$. A complex line in CP^2 represents this cycle, and in the projective

[3] CP^2 had a different interpretation in [13].

plane, two lines always intersect in one point. A copy of this cycle together with its normal neighbourhood can be interpreted as the proton part of the atom, whereas the neighbourhood of a point dual to this is interpreted as the electron. The neighbourhood of a point is just a 4-ball, with a 3-sphere boundary, but this is the same as in the Taub-NUT model of the electron, which is topologically just $\mathbb{R}^4$. The 3-sphere is a twisted circle bundle over a 2-sphere (the Hopf fibration) and this is sufficient to account for the electron charge.

In this paper, we have a novel proposal for the proton and neutron numbers. The 4-manifolds we consider are compact, to model neutral atoms. Our previous models always required charged particles to be non-compact so that the electric flux could escape to infinity, and this is an idea we will retain. We also restrict our manifolds to be complex algebraic surfaces, and their Chern numbers will be related to the proton and neutron numbers. There are more than enough examples to model all currently known isotopes of atoms. We will retain CP^2 as the model for the hydrogen atom.

2 Topology and Physics of Algebraic Surfaces

Complex surfaces [17] provide a rich supply of compact 4-manifolds. They are principally classified by two integer topological invariants, denoted c_1^2 and c_2. For a surface X, c_1 and c_2 are the Chern classes of the complex tangent bundle. c_2 is an integer because X has real dimension 4, whereas the (dual of the) canonical class c_1 is a particular 2-cycle in the second homology group, $H_2(X)$. c_1^2 is the intersection number of c_1 with itself, and hence another integer.

There are several other topological invariants of a surface X, but many are related to c_1^2 and c_2. Among the most fundamental are the Hodge numbers. These are the dimensions of the Dolbeault cohomology groups of holomorphic forms. In two complex dimensions the Hodge numbers are denoted $h^{i,j}$ with $0 \leq i, j \leq 2$. They are arranged in a Hodge diamond, as illustrated in figure 1. Serre duality, a generalisation of Poincaré duality, requires this diamond to be unchanged under a 180° rotation. For a connected surface, $h^{0,0} = h^{2,2} = 1$.

Complex algebraic surfaces are a fundamental subclass of complex surfaces [18, 19]. A complex algebraic surface can always be embedded in a complex projective space CP^n, and thereby acquires a Kähler metric from the ambient Fubini–Study metric on CP^n. For any Kähler manifold, the Hodge numbers have an additional symmetry, $h^{i,j} = h^{j,i}$. For a surface, this gives just one new relation, $h^{0,1} = h^{1,0}$. Not all complex surfaces are algebraic: some are still Kähler and satisfy this additional relation, but some are not

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & \frac{1}{2}b_1 & & \frac{1}{2}b_1 & \\
\frac{1}{2}\left(b_2^+ - 1\right) & & 1 + b_2^- & & \frac{1}{2}\left(b_2^+ - 1\right) \\
 & \frac{1}{2}b_1 & & \frac{1}{2}b_1 & \\
 & & 1 & &
\end{array}
$$

Figure 1: The Hodge diamond for a general complex surface (left) and its entries in terms of Betti numbers for an algebraic surface (right).

Kähler and do not satisfy it.

Particularly interesting for us are the holomorphic Euler number χ, which is an alternating sum of the entries on the top right (or equivalently, bottom left) diagonal of the Hodge diamond, and the analogous quantity for the middle diagonal, which we denote θ. More precisely,

$$\chi = h^{0,0} - h^{0,1} + h^{0,2}\,, \tag{2.1}$$
$$\theta = -h^{1,0} + h^{1,1} - h^{1,2}\,. \tag{2.2}$$

(Note the sign choice for θ.) The Euler number e and signature τ can be expressed in terms of these as

$$e = 2\chi + \theta\,, \tag{2.3}$$
$$\tau = 2\chi - \theta\,. \tag{2.4}$$

The first of these formulae reduces to the more familiar alternating sum of Betti numbers $e = b_0 - b_1 + b_2 - b_3 + b_4$, because each Betti number is the sum of the entries in the corresponding row of the Hodge diamond. The second formula is the less trivial Hodge index theorem. τ is more fundamentally defined by the splitting of the second Betti number into positive and negative parts, $b_2 = b_2^+ + b_2^-$. Over the reals the intersection form on the second homology group $H_2(X)$ is non-degenerate and can be diagonalised. b_2^+ is then the dimension of the positive subspace, and b_2^- the dimension of the negative subspace. The signature is $\tau = b_2^+ - b_2^-$.

The Chern numbers are related to χ and θ through the formulae

$$c_1^2 = 2e + 3\tau = 10\chi - \theta\,, \quad c_2 = e = 2\chi + \theta\,. \tag{2.5}$$

Their sum gives the Noether formula $\chi = \frac{1}{12}(c_1^2 + c_2)$, which is always integral.

For an algebraic surface, there are just three independent Hodge numbers and they are uniquely determined by the Betti numbers b_1, b_2^+ and b_2^-. The Hodge diamond must

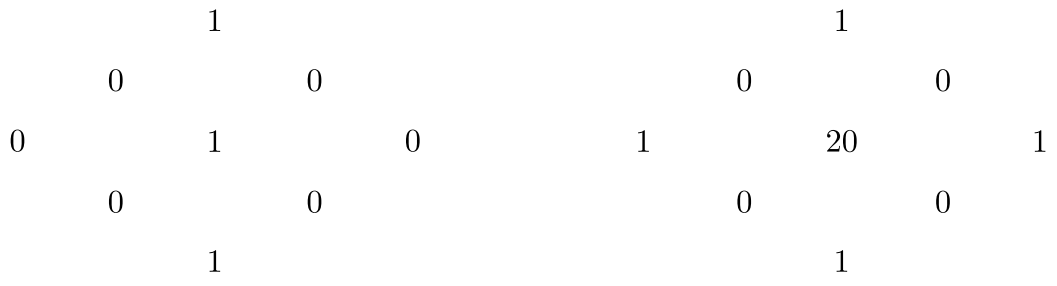

Figure 2: Hodge diamonds for the projective plane CP^2 (left) and for a K3 surface (right)

take the form shown on the right in figure 1, which gives the correct values for b_1, e and τ. Note that b_1 must be even and b_2^+ must be odd. χ and θ are now given by

$$\chi = \frac{1}{2}(1 - b_1 + b_2^+)\,, \tag{2.6}$$

$$\theta = 1 - b_1 + b_2^-\,. \tag{2.7}$$

If X is simply connected, which accounts for many examples, then $b_1 = 0$. Hodge diamonds for the projective plane CP^2 and for a K3 surface, both of which are simply connected, are shown in figure 2. For the projective plane $\chi = 1$ and $\theta = 1$, so $e = 3$ and $\tau = 1$, and for a K3 surface $\chi = 2$ and $\theta = 20$, so $e = 24$ and $\tau = -16$.

Our proposal is to model neutral atoms by complex algebraic surfaces and to interpret χ as proton number P, and θ as baryon number B. So neutron number is $N = \theta - \chi$. This proposal fits with CP^2 having $P = 1$ and $N = 0$. We will see later that for each positive value of P there is an interesting, finite range of allowed N values.

In terms of e and τ,

$$P = \frac{1}{4}(e + \tau)\,, \quad B = \frac{1}{2}(e - \tau)\,, \quad N = \frac{1}{4}(e - 3\tau)\,. \tag{2.8}$$

Note that for a general, real 4-manifold, these formulae for P and N might be fractional, and would need modification. It is also easy to verify that in terms of P and N,

$$c_1^2 = 9P - N\,, \tag{2.9}$$

$$c_2 = e = 3P + N\,, \tag{2.10}$$

$$\tau = P - N\,. \tag{2.11}$$

The simple relation of signature τ to the difference between proton and neutron numbers is striking. If we write $N = P + N_{\mathrm{exc}}$, where N_{exc} denotes the excess of neutrons over protons (which is usually zero or positive, but can be negative), then $\tau = -N_{\mathrm{exc}}$.

If an algebraic surface X is simply connected then $b_1 = 0$, and in terms of P and N,

$$b_2^+ = 2P - 1\,, \quad b_2^- = P + N - 1 = 2P - 1 + N_{\rm exc}\,. \tag{2.12}$$

These formulae will be helpful when we consider intersection forms in more detail.

The class of surfaces that we will use, as models of atoms, are those with c_1^2 and c_2 non-negative. Many of these are minimal surfaces of general type. Perhaps the most important results on the geometry of algebraic surfaces are certain inequalities that the Chern numbers of minimal surfaces of general type have to satisfy. The basic inequalities are that c_1^2 and c_2 are positive. Also, there is the Bogomolov–Miyaoka–Yau (BMY) inequality which requires $c_1^2 \leq 3c_2$, and finally there is the Noether inequality $5c_1^2 - c_2 + 36 \geq 0$. These inequalities can be converted into the following inequalities on P and N:

$$P > 0\,, \quad 0 \leq N < 9P\,, \quad N \leq 7P + 6\,. \tag{2.13}$$

All integer values of P and N satisfying these are allowed. The allowed region is shown in figure 3, and corresponds to the allowed region shown on page 229 of [17], or in the article [20].

There are also the elliptic surfaces (including the Enriques surface and K3 surface) where $c_1^2 = 0$ and c_2 is non-negative, and we shall include these among our models. Here, $P \geq 0$ and $N = 9P$, so $c_2 = 12P$ and $\tau = -8P$. CP^2 is also allowed, even though it is rational and not of general type, because c_1^2 and c_2 are positive. In addition to CP^2, there are further surfaces on the BMY line $c_1^2 = 3c_2$ [21], which have $P > 1$ and $N = 0$.

Physicists usually denote an isotope by proton number and baryon number, where proton number P is determined by the chemical name, and baryon number is $P + N$. For example, the notation ^{56}Fe means the isotope of iron with $P = 26$ and $N = 30$. The currently recognised isotopes are shown in figure 4.

The shape of the allowed region of algebraic surfaces qualitatively matches the region of recognised isotopes, and this is the main justification for our proposal. For example, for $P = 1$, the geometric inequalities allow N to take values from 0 up to 9. This corresponds to a possible range of hydrogen isotopes from ^{1}H to ^{10}H. Physically, the well-known hydrogen isotopes are the proton, deuterium and tritium, that is, ^{1}H, ^{2}H and ^{3}H respectively, but nuclear physicists recognise isotopes of a quasi-stable nature (resonances) up to ^{7}H, with $N = 6$.

The minimal models for the common isotopes, the proton alone, and deuterium, each bound to one electron, are CP^2 and the complex quadric surface Q. The quadric is the product $\mathrm{Q} = \mathrm{CP}^1 \times \mathrm{CP}^1$, with $e = 4$ and $\tau = 0$. We shall say more about its intersection form below.

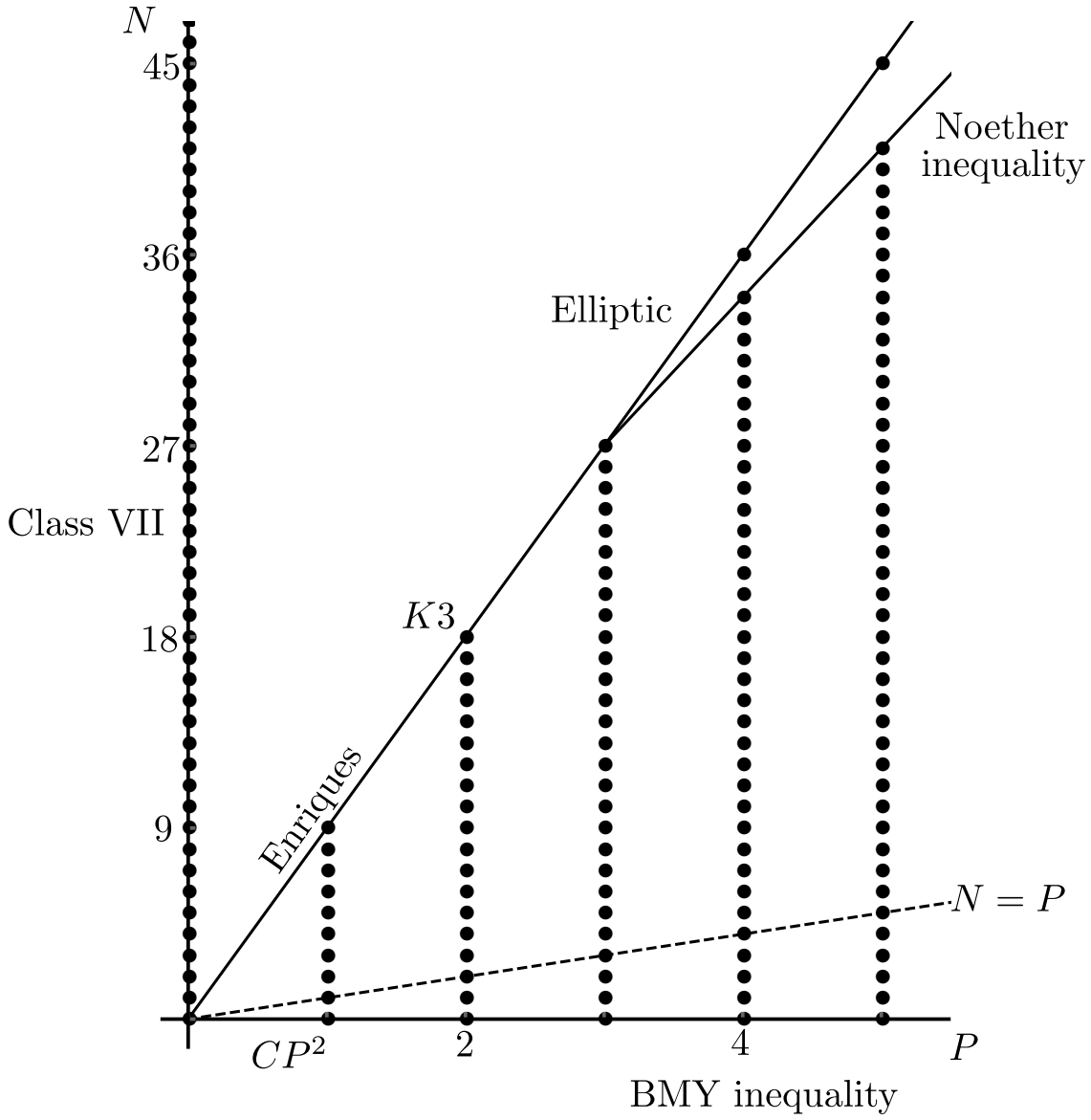

Figure 3: The allowed region of proton numbers P, and neutron numbers N, for atoms modelled as algebraic surfaces. The boundaries correspond to inequalities on the Chern numbers. Note the change of slope from 9 to 7 at the point with $P = 3$, $N = 27$. The line $N = P$ corresponds to surfaces with zero signature ($\tau = 0$).

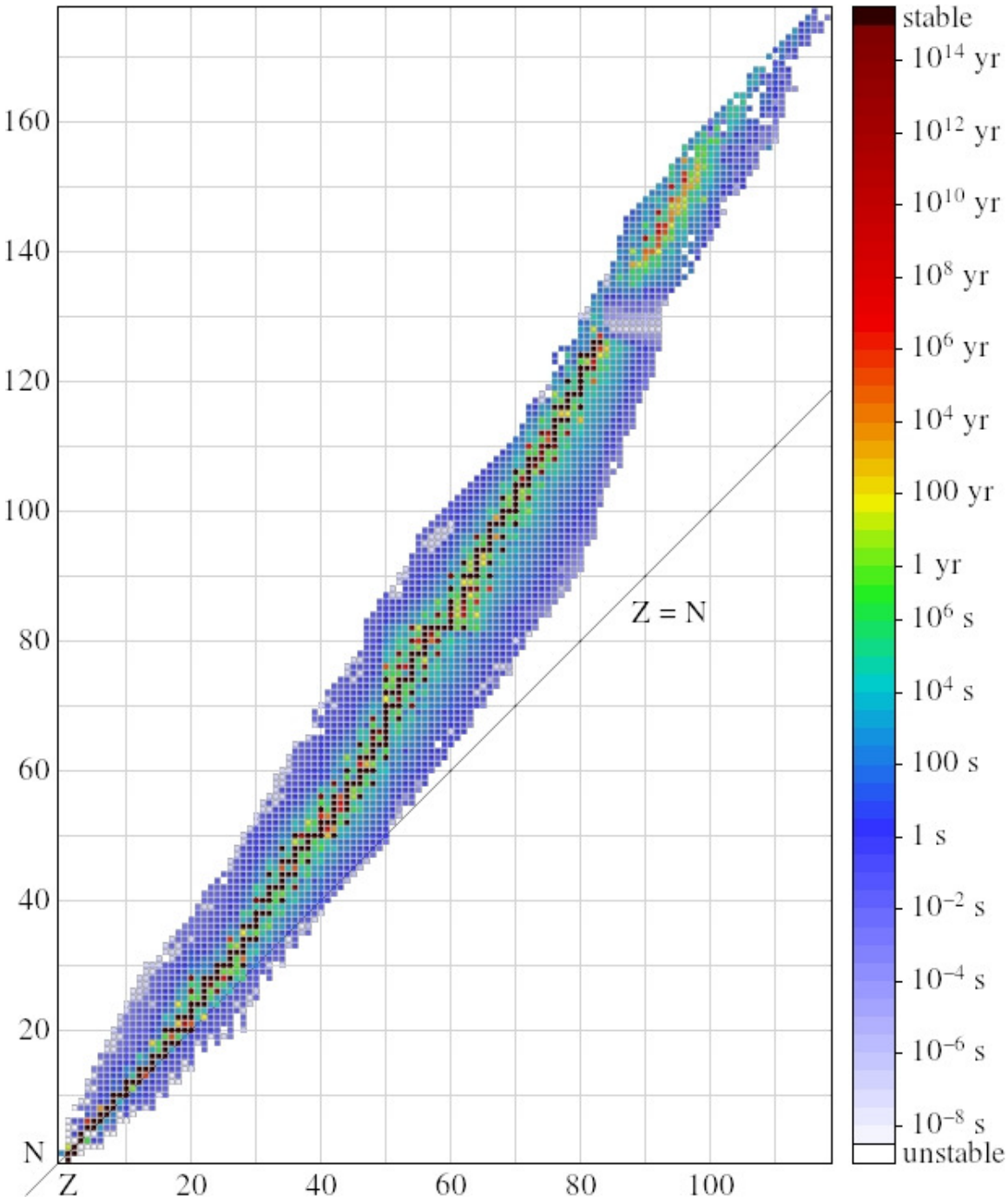

Figure 4: Nuclear isotopes. The horizontal axis is proton number P (Z in physics notation) and the vertical axis is neutron number N. The shading (colouring online) indicates the lifetime of each isotope, with black denoting stability (infinite lifetime).

For $P = 2$, N is geometrically allowed in the range 0 to 18. The corresponding algebraic surfaces should model helium isotopes from ^{2}He to ^{20}He. Isotopes from ^{3}He up to ^{10}He are physically recognised. All these potentially form neutral atoms with two electrons. The helium isotope with no neutrons is not listed in some nuclear tables, but there does exist an unbound diproton resonance, and diprotons are sometimes emitted when heavier nuclei decay. The most common, stable helium isotope is ^{4}He, with two protons and two neutrons, but ^{3}He is also stable. ^{4}He nuclei are also called alpha-particles, and play a key role in nuclear processes and nuclear structure. It is important to have a good geometrical model of an alpha-particle, which ideally should match the cubically symmetric $B = 4$ Skyrmion that is a building block for many larger Skyrmions [22, 23, 8, 10].

3 Valley of Stability

Running through the nuclear isotopes is the valley of stability [2]. In figure 4, this is the irregular curved line of stable nuclei marked in black. On either side, the nuclei are unstable, with lifetimes of many years near the centre of the valley, reducing to microseconds further away. Sufficiently far from the centre are the nuclear drip lines, where a single additional proton or neutron has no binding at all, and falls off in a time of order 10^{-23} seconds.

For small nuclei, for P up to about 20, the valley is centred on the line $N = P$. In the geometrical model, this line corresponds to surfaces with signature $\tau = 0$. For larger P, nuclei in the valley have a neutron excess, $N_{\rm exc}$, which increases slowly from just a few when P is near 20 to over 50 for the quasi-stable uranium isotopes with $P = 92$, and slightly more for the heaviest artifically produced nuclei with P approaching 120.

In standard nuclear models, the main effect explaining the valley is the Pauli principle. Protons and neutrons have a sequence of rather similar 1-particle states of increasing energy, and just one particle can be in each state. For given baryon number, the lowest-energy state has equal proton and neutron numbers, filling the lowest available states. If one proton is replaced by one neutron, the proton state that is emptied has lower energy than the neutron state that is filled, so the total energy goes up. An important additional effect is a pairing energy that favours protons to pair up and neutrons to pair up. Most nuclei with P and N both odd are unstable as a result.

For larger values of P, the single-particle proton energies tend to be higher than the single-particle neutron energies, because in addition to the attractive, strong nuclear forces which are roughly the same for protons and neutrons, there is the electrostatic Coulomb

repulsion that acts between protons alone. This effect becomes important for nuclei with large P, and favours neutron-rich nuclei. It also explains the instability of all nuclei with P larger than 83. These nuclei simply split up into smaller nuclei, either by emitting an alpha-particle, or by fissioning into larger fragments. However, the lifetimes can be billions of years in some cases, which is why uranium, with $P = 92$, is found in nature in relatively large quantities.

Note that if $N = P$, then the electric charge is half the baryon number, and according to formula (1.1), the third component of isospin is zero. By studying nuclear ground states and excited states, one can determine the complete isospin, and it is found to be minimal for stable nuclei. So nuclei with $N = P$ have zero isospin. When the baryon number is odd, the most stable nuclei have N just one greater than P (if P is not too large), and the isospin is $\frac{1}{2}$. Within the Skyrme model, isospin arises from the quantisation of internal degrees of freedom, associated with an SO(3) symmetry acting on the pion fields. There is an energy contribution proportional to the squared isospin operator $\mathbf{I}^2$, analogous to the spin energy proportional to $\mathbf{J}^2$. In the absence of Coulomb effects, the energy is minimised by fixing the isospin to be zero or $\frac{1}{2}$. The Coulomb energy competes with isospin, and shifts the total energy minimum towards neutron-rich nuclei.

These are the general trends of nuclear energies and lifetimes. However there is a lot more in the detail. Each isotope has its own character, depending on its proton and neutron numbers. This is most clear in the energy spectra of excited states, and the spins of the ground and excited states. Particularly interesting is the added stability of nuclei where either the proton or neutron number is magic. The smaller magic numbers are 2, 8, 20, 28, 50. It is rather surprising that protons and neutrons can be treated independently with regard to the magic properties. This appears to contradict the importance of isospin, in which protons and neutrons are treated as strongly influencing each other.

Particularly stable nuclei are those that are doubly magic, like ^{4}He, ^{16}O, ^{40}Ca and ^{48}Ca. ^{40}Ca is the largest stable nucleus with $N = P$. ^{48}Ca is also stable, and occurs in small quantities in nature, but is exceptionally neutron-rich for a relatively small nucleus.

The important issue for us here is to what extent our proposed geometrical model based on algebraic surfaces is compatible with these nuclear phenomena, not forgetting the electron structure in a neutral atom. There are some broad similarities. First there is the "geography" of surfaces we have discussed above, implying that the geometrical inequalities restrict the range of neutron numbers. Algebraic geometers also refer to "botany", the careful construction and study of surfaces with particular topological invariants. The patterns are very complicated. Some surfaces are simple to construct, others less so, and their internal structure is very variable. This is analogous to the complications of the nuclear landscape, and the similar complications (better understood) of the electron orbitals and atomic shell structure.

Rather remarkable is that the line of nuclear stability where $N = P$ corresponds to the simple geometrical condition that the signature τ is zero. We have not yet tried to pinpoint an energy function on the space of surfaces, but clearly it would be easy to include a dominant contribution proportional to τ^2, whose minimum would be in the desired place. Mathematicians have discovered that it is much easier to construct surfaces on this line, and on the neutron-rich side of it, where τ is negative, than on the proton-rich side. There are always minimal surfaces on the neutron-rich side which are simply connected, but not everywhere on the proton-rich side. The geometry of surfaces therefore distinguishes protons from neutrons rather clearly. This is attractive for the physical interpretation, as it can be regarded as a prediction of an asymmetry between the proton and neutron. In standard nuclear physics it is believed that in an ideal world with no electromagnetic effects, there would be an exact symmetry between the proton and neutron, but in reality they are not the same, partly because of Coulomb energy, but more fundamentally, because their constituent up (u) and down (d) quarks are not identical in their masses, making the proton (uud) less massive than the neutron (udd), despite its electric charge.

The geometrical model would need an energy contribution that favours neutrons over protons for the larger nuclei and atoms. One possibility has been explored by LeBrun [24, 25]. This is the infimum, over complex surfaces with given topology, of the L^2 norm of the scalar curvature. For surfaces with b_1 even, including all surfaces that are simply connected, this infimum is simply a constant multiple of c_1^2. The scalar curvature can be zero for surfaces on the line $c_1^2 = 0$, for example the K3 surface, which is the extreme of neutron-richness, with $P = 2$ and $N = 18$. It would be interesting to consider more carefully the energy landscape for an energy that combines τ^2 and a positive multiple of the L^2 norm of scalar curvature.

4 Intersection Form

A complex surface X is automatically oriented, so any pair of 2-cycles has an unambiguous intersection number [26]. Given a basis α^i of 2-cycles for the second homology group $H_2(X)$, the matrix Ω^{ij} of intersection numbers is called the intersection form of X. $\Omega^{ij} \equiv \Omega(\alpha^i, \alpha^j)$ is the intersection number of basis cycles α^i and α^j, and the self-intersection number Ω^{ii} is the intersection number of α^i with a generic smooth deformation of itself. Ω is a symmetric matrix of integers, and by Poincaré duality it is unimodular (of determinant ± 1). Over the reals, such a symmetric matrix is diagonalisable, and the diagonal entries are either $+1$ or -1. The numbers of each of these are b_2^+ and b_2^-, respectively, and we have already given an interpretation of them for simply-connected algebraic surfaces X in terms of P and N in equation (2.12) above.

However, diagonalisation over the reals does not make sense for cycles, because one can end up with fractional cycles in the new basis. One may only change the basis of cycles using an invertible matrix of integers, whose effect is to conjugate Ω by such a matrix. The classification of intersection forms is finer over the integers than the reals.

For almost all algebraic surfaces, Ω is indefinite. b_2^+ is always positive, and b_2^- is positive too, except for surfaces with $b_1 = 0$ and $B = \theta = 1$. So the only surfaces for which the intersection form Ω is definite are CP^2, and perhaps additionally the fake projective planes, for which we have not found a physical interpretation. For CP^2, with $P = 1$ and $N = 0$, the intersection form is the 1×1 matrix $\Omega = (1)$.

Non-degenerate, indefinite forms over the integers have a rather simple classification. The basic dichotomy is between those that are odd and those that are even. An odd form is one for which at least one entry Ω^{ii} is odd, or more invariantly, $\Omega(\alpha, \alpha)$ is odd for some 2-cycle α. An odd form can always be diagonalised, with entries $+1$ and -1 on the diagonal.

Even forms are more interesting. Here $\Omega(\alpha, \alpha)$ is even for any cycle α. The simplest example is

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{4.1}$$

This is the intersection form of the quadric Q, with the two CP^1 factors as basis cycles, α^1 and α^2. If $\alpha = x\alpha^1 + y\alpha^2$ then $\Omega(\alpha, \alpha) = 2xy$, so is always even. Over the reals this form can be diagonalised and has entries $+1$ and -1 (the eigenvalues). So it has zero signature. But the diagonalisation involves fractional matrices, and is not possible over the integers. The intersection form (4.1) is called the "hyperbolic plane". A second ingredient in even intersection forms is the matrix $-E_8$. This is the negative of the Cartan matrix of the Lie algebra E_8 (with diagonal entries -2). It is even and unimodular. By itself this form is negative definite, but when combined with hyperbolic plane components, the result is indefinite, as needed. The most general (indefinite) even intersection form for an algebraic surface can be brought to the block diagonal form

$$\Omega = l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + m(-E_8) , \tag{4.2}$$

with $l > 0$ and $m \geq 0$. l must be odd, and the Betti numbers are $b_2^+ = l$ and $b_2^- = l + 8m$. The signature is $\tau = -8m$.

For most surfaces, the signature is not a multiple of 8, so the intersection form is odd. If the signature is a multiple of 8, it may be even. For given Betti numbers, there could be two distinct minimal surfaces (or families of these), one with an odd intersection form, and the other with an even intersection form. We do not know if surfaces with both types of intersection form always occur.

We can reexpress these conditions in terms of the physical numbers P and N. If $N_{\text{exc}} = N - P$ is neither zero nor a positive multiple of 8, then the intersection form must be odd. If $N = P$, then the intersection form can be of the hyperbolic plane type $l\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with $l = 2P - 1$, or it might still be odd. Notice that l is odd, as it must be. The isotopes for which even intersection forms are possible therefore include all those with $N = P$. These are numerous. In addition to the stable isotopes with $N = P$ that occur up to ^{40}Ca, with $P = 20$, there are many that are quasi-stable, like ^{52}Fe, with $P = 26$. The heaviest recognised isotopes with $N = P$ are ^{100}Sn and perhaps ^{108}Xe, with $P = 50$ and $P = 54$. Our geometrical model suggests that the additional stability of these isotopes is the result of the nontrivial structure of an even intersection form.

If $N_{\text{exc}} = 8m$ then the intersection form can be of type (4.2), again with $l = 2P - 1$, but it might also be odd. Examples are the Enriques surface, for which $l = 1$ and $m = 1$, and the K3 surface, for which $l = 3$ and $m = 2$. The potential isotopes corresponding to these surfaces are ^{10}H and ^{20}He. These are both so neutron-rich that they have not been observed, but there are many heavier nuclei (and corresponding atoms) for which the neutron excess is a multiple of 8.

There is some evidence that nuclei whose neutron excess is a multiple of 8 have additional stability. The most obvious example is ^{48}Ca, but this is conventionally attributed to the shell model, as $P = 20$ and $N = 28$, both magic numbers. A more interesting and less understood example is the heaviest known isotope of oxygen, ^{24}O, with a neutron excess of 8. This example and others do not obviously fit with the shell model. The most stable isotope of iron is ^{56}Fe, whose neutron excess is 4, but it is striking that ^{60}Fe, whose neutron excess is 8, has a lifetime of over a million years. Here $P = 26$ and $N = 34$. ^{64}Ni, also with a neutron excess of 8, is one of the stable isotopes of nickel. There are also striking examples of stable or relatively stable isotopes with neutron excesses of 16 or 24. Some of these are outliers compared to the general trends in the valley of stability. An example is ^{124}Sn, the heaviest stable isotope of tin, with $N_{\text{exc}} = 24$. A more careful study would be needed to confirm if the additional stability of isotopes whose neutron excess is a multiple of 8 is statistically significant.

There is no evidence that a neutron deficit of 8 has a stabilising effect. In fact, almost no nuclei with such a large neutron deficit are recognised. The only candidate is ^{48}Ni, with the magic numbers $P = 28$ and $N = 20$.

5 Other Surfaces

In addition to the minimal surfaces of general type there are various other classes of algebraic surface. Do these have a physical interpretation?

On a surface X it is usually possible to "blow up" one or more points. The result is not minimal, because a minimal surface, by definition, is one that cannot be constructed by blowing up points on another surface. Blowing up one point increases c_2 by 1 and decreases c_1^2 by 1. This is equivalent, in our model, to increasing N by 1, leaving P unchanged. In other words, one neutron has been added. Topologically, blowing up is a local process, equivalent to attaching (by connected sum) a copy of $\overline{\mathrm{CP}^2}$. This adds a 2-cycle that has self-intersection -1, but no intersection with any other 2-cycle. The rank (size) of the intersection form Ω increases by 1, with an extra -1 on the diagonal, and the remaining entries of the extra row and column all zero. This automatically makes the intersection form odd, so any previously even form now becomes diagonalisable.

The physical interpretation seems to be that a neutron has been added, well separated from any other neutron or proton. This adds a relatively high energy, more than if the additional neutron were bound into an existing nucleus. Minimal algebraic surfaces, and especially those with even intersection forms, should correspond to tightly bound nuclei and atoms, having lower energy.

The simplest example is the blow up of one point on CP^2. The result is the Hirzebruch surface H_1, which is a non-trivial CP^1 bundle over CP^1. Its intersection form is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Hirzebruch surface and quadric are both simply connected and have the same Betti numbers, $b_2^+ = b_2^- = 1$, corresponding to $P = 1$ and $N = 1$, but the intersection form is odd for the Hirzebruch surface and even for the quadric. The proposed interpretation is that the Hirzebruch surface represents a separated proton, neutron and electron, whereas the quadric represents the deuterium atom, with a bound proton and neutron as its nucleus, orbited by the electron.

There is an inequality of LeBrun for the L^2 norm of the Ricci curvature supporting this interpretation [24, 25]. The norm increases if points on a minimal surface are blown up, the increase being a constant multiple of the number of blown-up points. This strongly indicates that the norm of the Ricci curvature and the norm of the scalar curvature, possibly with different coefficients, should be ingredients in the physical energy.

So far, we have not considered any surfaces X that could represent a single neutron, or a cluster of neutrons. Candidates are the surfaces of Type VII. These have $c_1^2 = -c_2$, with c_2 positive, equivalent to $P = 0$ and arbitrary positive N. These surfaces are complex, but

are not algebraic and do not admit a Kähler metric. They are also not simply connected. It is important to have a model of a single neutron. The discussion of blow-ups suggests that $\overline{\mathrm{CP}^2}$ is another possible model. In this case a single neutron would be associated with a 2-cycle with self-intersection -1, mirroring the proton inside CP^2 being represented by a 2-cycle with self-intersection $+1$.

A free neutron is almost stable, having a lifetime of approximately 10 minutes. There is considerable physical interest in clusters of neutrons. There is a dineutron resonance similar to the diproton resonance. Recently there has been some experimental evidence for a tetraneutron resonance, indicating some tendency for four neutrons to bind [27]. Octaneutron resonances have also been discussed, but no conclusive evidence for their existence has yet emerged. Neutron stars consist of multitudes of neutrons, accompanied perhaps by a small number of other particles (protons and electrons), but their stability is only possible because of the gravitational attraction supplementing the nuclear forces. Standard Newtonian gravity is of course negligible for atomic nuclei.

Products of two Riemann surfaces (algebraic curves) of genus 2 or more are examples of minimal surfaces of general type, but they are certainly not simply connected. Their interpretation as atoms should be investigated. Other surfaces, for example ruled surfaces, may have some physical interpretation, but our formulae would give them negative proton and neutron numbers. They do not model antimatter, that is, combinations of antiprotons, antineutrons and positrons, because antimatter is probably best modelled using the complex conjugates of surfaces modelling matter. Also bound states of protons and antineutrons, with positive P and negative N, do not seem to exist.

6 Conclusions

We have proposed a new geometrical model of matter. It goes beyond our earlier proposal [13] in that it can accommodate far more than just a limited set of basic particles. In principle, the model can account for all types of neutral atom.

Each atom is modelled by a compact, complex algebraic surface, which as a real manifold is four-dimensional. The physical quantum numbers of proton number P (equal to electron number for a neutral atom) and neutron number N are expressed in terms of the Chern numbers c_1^2 and c_2 of the surface, but they can also be expressed in terms of combinations of the Hodge numbers, or of the Betti numbers b_1, b_2^+ and b_2^-.

Our formulae for P and N were arrived at by considering the interpretation of just a few examples of algebraic surfaces – the complex projective plane CP^2, the quadric surface Q, and the Hirzebruch surface H_1. Some consequences, which follow from known

constraints on algebraic surfaces, can therefore be regarded as predictions of the model. Among these are that P is any positive integer, and that N is bounded below by 0 and bounded above by the lesser of $9P$ and $7P + 6$. This encompasses all known isotopes. A most interesting prediction is that the line $N = P$, which is the centre of the valley of nuclear stability for small and medium-sized nuclei, corresponds to the the line $\tau = 0$, where $\tau = b_2^+ - b_2^-$ is the signature. Surfaces with τ positive and τ negative are known to be qualitatively different, which implies that in our model there is a qualitative difference between proton-rich and neutron-rich nuclei.

For simply connected surfaces with $b_1 = 0$ (or more generally, if b_1 is held fixed) then an increase of P by 1 corresponds to an increase of b_2^+ by 2. The interpretation is that there are two extra 2-cycles with positive self-intersection, corresponding to the extra proton and the extra electron. This matches our earlier models, where a proton was associated with such a 2-cycle [13], and where multi-Taub-NUT space with n NUTs modelled n electrons [15, 16]. On the other hand, an increase of N by 1 corresponds to an increase of b_2^- by 1. This means that a neutron is associated with a 2-cycle of negative self-intersection, which differs from our earlier ideas, where a neutron was modelled by a 2-cycle with zero self-intersection. It appears now that the intersection numbers are related to isospin (whose third component is $\frac{1}{2}$ for a proton and $-\frac{1}{2}$ for a neutron) rather than to electric charge (1 for a proton and 0 for a neutron).

Clearly, much further work is needed to develop these ideas into a physical model of nuclei and atoms. We have earlier made a few remarks about possible energy functions for complex surfaces. Some combination of the topological invariants and non-topological curvature integrals could be explored, and compared with the detailed information on the energies of nuclei and atoms in their ground states. It will be important to account for the quantum mechanical nature of the ground and excited states, their energies and spins. Discrete energy gaps could arise from discrete changes in geometry, for example, by replacing a blown-up surface with a minimal surface, or by considering the effect of changing b_1 while keeping P and N fixed, or by comparing different embeddings of an algebraic surface in (higher-dimensional) projective space. In some cases there should be a discrete choice for the intersection form. There are also possibilities for finding an analogue of a Schrödinger equation using linear operators, like the Laplacian or Dirac operator, acting on forms or spinors on a surface. Alternatively, the right approach may be to consider the continuous moduli of surfaces as dynamical variables, and then quantise these. Some of the ideas just mentioned have already been investigated in the context of single particles, modelled by the Taub-NUT space or another non-compact 4-manifold [14, 28]. Further physical processes, for example, the fission of larger nuclei, and the binding of atoms into molecules, also need to be addressed.

Before these investigations can proceed, it will be necessary to decide what metric

structure the surfaces need. Previously, we generally required manifolds to have a self-dual metric, i.e. to be gravitational instantons, but this now seems too rigid, as there are very few compact examples. Requiring a Kähler–Einstein metric may be more reasonable, although these do not exist for all algebraic surfaces [29, 30].

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References

[1] W. Thomson, On vortex atoms, *Trans. R. Soc. Edin.* **6** (1867) 94.

[2] J. Lilley, *Nuclear Physics: Principles and Applications*, Chichester: Wiley, 2001.

[3] F. Rohrlich, *Classical Charged Particles (3rd ed.)*, Singapore: World Scientific, 2007.

[4] T.H.R. Skyrme, A nonlinear field theory, *Proc. Roy. Soc. Lond.* **A260** (1961) 127; A unified field theory of mesons and baryons, *Nucl. Phys.* **31** (1962) 556.

[5] G.S. Adkins, C.R. Nappi and E. Witten, Static properties of nucleons in the Skyrme model, *Nucl. Phys.* **B228** (1983) 552.

[6] D.H. Perkins, *Introduction to High Energy Physics (4th ed.)*, Cambridge: CUP, 2000.

[7] R.A. Battye et al., Light nuclei of even mass number in the Skyrme model, *Phys. Rev.* **C80** (2009) 034323.

[8] P.H.C. Lau and N.S. Manton, States of Carbon-12 in the Skyrme model, *Phys. Rev. Lett.* **113** (2014) 232503.

[9] C.J. Halcrow, Vibrational quantisation of the $B = 7$ Skyrmion, *Nucl. Phys.* **B904** (2016) 106.

[10] C.J. Halcrow, C. King and N.S. Manton, A dynamical α-cluster model of ^{16}O, arXiv:1608.05048 (2016).

[11] D. Finkelstein and J. Rubinstein, Connection between spin, statistics and kinks, *J. Math. Phys.* **9** (1968) 1762.

[12] M. Atiyah and N.S. Manton, Skyrmions from instantons, *Phys. Lett.* **B222** (1989) 438.

[13] M. Atiyah, N.S. Manton and B.J. Schroers, Geometric models of matter, *Proc. Roy. Soc. Lond.* **A468** (2012) 1252.

[14] R. Jante and B.J. Schroers, Dirac operators on the Taub-NUT space, monopoles and SU(2) representations, *JHEP* 01 (2014) 114.

[15] G. Franchetti and N.S. Manton, Gravitational instantons as models for charged particle systems, *JHEP* 03 (2013) 072.

[16] G. Franchetti, Harmonic forms on ALF gravitational instantons, *JHEP* 12 (2014) 075.

[17] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Berlin, Heidelberg: Springer, 1984.

[18] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, New York, Chichester: Wiley Classics, 1994.

[19] C. Voisin, *Hodge Theory and Complex Algebraic Geometry, Vol. I*, Cambridge: CUP, 2002.

[20] *Enriques–Kodaira classification*, Wikipedia, 2016.

[21] D.I. Cartwright and T. Steger, Enumeration of the 50 fake projective planes, *C. R. Acad. Sci. Paris, Ser. I* **348** (2010) 11.

[22] E. Braaten, S. Townsend and L. Carson, Novel structure of static multisoliton solutions in the Skyrme model, *Phys. Lett.* **B235** (1990) 147.

[23] R.A. Battye, N.S. Manton and P.M. Sutcliffe, Skyrmions and the α-particle model of nuclei, *Proc. Roy. Soc. Lond.* **A463** (2007) 261.

[24] C. LeBrun, Four-manifolds without Einstein metrics, *Math. Res. Lett.* **3** (1996) 133.

[25] C. LeBrun, Ricci curvature, minimal volumes, and Seiberg–Witten theory, *Invent. Math.* **145** (2001) 279.

[26] S.K. Donaldson and P.B. Kronheimer, *Geometry of Four-Manifolds*, Oxford: OUP, 1990.

[27] K. Kisamori et al., Candidate resonant tetraneutron state populated by the ^{4}He(^{8}He, ^{8}Be) reaction, *Phys. Rev. Lett.* **116** (2016) 052501.

[28] R. Jante and B.J. Schroers, Spectral properties of Schwarzschild instantons, arxiv:1604.06080.

[29] T. Ochiai (ed.), *Kähler Metric and Moduli Spaces: Advanced Studies in Pure Mathematics 18-II*, Tokyo: Kinokuniya, 1990.

[30] G. Tian, Kähler–Einstein metrics with positive scalar curvature, *Invent. Math.* **137** (1997) 1.