This shows that $\int_{F_p} j_p{}^*\omega$ is independent of p, for $p \in A$. Using connectedness, it is easy to see that it is independent of p for all $p \in M$, so we will denote it simply by $\int_F j^*\omega$. Thus

$$\int_{\pi^{-1}(A)} \pi^*\eta \wedge \omega = \int_A \pi^*\eta \cdot \int_F j^*\omega.$$

Comparing with equation (1), and utilizing partitions of unity, we conclude that

$$\int_F j^*\omega = 1,$$

which proves the first part of the theorem.

Now suppose we have another class $U' \in H_c^k(E)$. Since

$$H_c^k(E) \approx H^n(E) \approx H^n(M) \approx \mathbb{R},$$

it follows that $U' = cU$ for some $c \in \mathbb{R}$. Consequently,

$$j_p{}^*U' = j_p{}^*cU = c \cdot \nu_p.$$

Hence U' has the same property as U only if $c = 1$. ❖

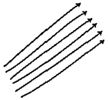
The Thom class U of $\xi = \pi : E \to M$ can now be used to determine an element of $H^k(M)$. Let $s : M \to E$ be any section; there always is one (namely, the 0-section) and any two are clearly smoothly homotopic. We define the **Euler class** $\chi(\xi) \in H^k(M)$ of ξ by

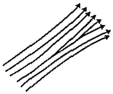
$$\chi(\xi) = s^*U.$$

Notice that if ξ has a non-zero section $s : M \to E$, and $\omega \in C_c^k(E)$ represents U, then a suitable multiple $c \cdot s$ of s takes M to the complement of support ω. Hence, in this case

$$\chi(\xi) = (c \cdot s)^*U = 0.$$

The terminology "Euler class" is connected with the special case of the bundle TM, whose sections are, of course, vector fields on M. If X is a vector field on M which has an isolated 0 at some point p (that is, $X(p) = 0$, but $X(q) \neq 0$ for $q \neq p$ in a neighborhood of p), then, quite independently of our previous considerations, we can define an "index" of X at p. Consider first a vector field X on an open set $U \subset \mathbb{R}^n$ with an isolated zero at $0 \in U$. We can define a function $f_X : U - \{0\} \to S^{n-1}$ by $f_X(p) = X(p)/|X(p)|$. If $i : S^{n-1} \to U$ is $i(p) = \varepsilon p$, mapping S^{n-1} into U, then the map $f_X \circ i : S^{n-1} \to S^{n-1}$ has a certain degree; it is independent of ε, for small ε, since the maps $i_1, i_2 : S^{n-1} \to U$ corresponding to ε_1 and ε_2 will be smoothly homotopic. This degree is called the **index** of X at 0.


index 0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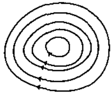
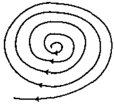
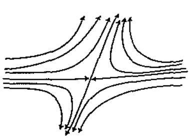
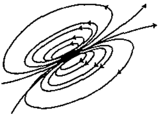
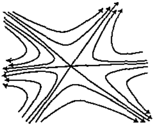
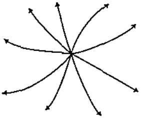

index 1

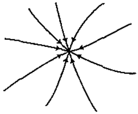

index 1


index -1


index 2


index -2


index 1 in $\mathbb{R}^n$


index $(-1)^n$ in $\mathbb{R}^n$

Now consider a diffeomorphism $h : U \to V \subset \mathbb{R}^n$ with $h(0) = 0$. Recall that h_*X is the vector field on V with

$$(h_*X)(y) = h_*(X_{h^{-1}(y)}).$$

Clearly 0 is also an isolated zero of h_*X.

27. LEMMA. If $h : U \to V \subset \mathbb{R}^n$ is a diffeomorphism with $h(0) = 0$, and X has an isolated 0 at 0, then the index of h_*X at 0 equals the index of X at 0.

PROOF. Suppose first that h is orientation preserving. Define

$$H: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

by

$$H(x,t) = \begin{cases} h(tx) & 0 < t \leq 1 \\ Dh(0)(x) & t = 0. \end{cases}$$

This is a smooth homotopy; to prove that it is smooth at 0 we use Lemma 3-2 (compare Problem 3-32). Each map $H_t = x \mapsto H(x,t)$ is clearly a diffeomorphism, $0 \leq t \leq 1$. Note that $H_1 \in \mathrm{SO}(n)$, since h is orientation preserving. There is also a smooth homotopy $\{H_t\}$, $1 \leq t \leq 2$ with each $H_t \in \mathrm{SO}(n)$ and $H_2 =$ identity, since $\mathrm{SO}(n)$ is connected. So (see Problem 8-25), the map h is smoothly homotopic to the identity, via maps which are diffeomorphisms. This shows that f_{h_*X} is smoothly homotopic to f_X on a sufficiently small region of $\mathbb{R}^n - \{0\}$. Hence the degree of $f_{h_*X} \circ i$ is the same as the degree of $f_X \circ i$.

To deal with non-orientation preserving h, it obviously suffices to check the theorem for $h(x) = (x^1, \dots, x^{n-1}, -x^n)$. In this case

$$f_{h_*X} = h \circ f_X \circ h^{-1},$$

which shows that degree $f_{h_*X} \circ i =$ degree $f_X \circ i$. ❖

As a consequence of Lemma 27, we can now define the index of a vector field on a manifold. If X is a vector field on a manifold M, with an isolated zero at $p \in M$, we choose a coordinate system (x, U) with $x(p) = 0$, and define the index of X at p to be the index of x_*X at 0.

28. THEOREM. Let M be a compact connected manifold with an orientation μ, which is, by definition, also an orientation for the tangent bundle $\xi = \pi: TM \to M$. Let $X: M \to TM$ be a vector field with only a finite number of zeros, and let σ be the sum of the indices of X at these zeros. Then

$$\chi(\xi) = \sigma \cdot \mu \in H^n(M).$$

PROOF. Let $p_1, \dots, p_r$ be the zeros of X. Choose disjoint coordinate systems $(U_1, x_1), \dots, (U_r, x_r)$ with $x_i(p_i) = 0$, and let

$$B_i = x_i^{-1}(\{p \in \mathbb{R}^n : |p| \leq 1\}).$$

If $\omega \in C_c^n(E)$ is a closed form representing the Thom class U of ξ, then we are trying to prove that

$$\int_{(M,\mu)} X^*(\omega) = \sigma.$$

We can clearly suppose that $X(q) \notin$ support ω for $q \notin \bigcup_i B_i$. So

$$\int_M X^*(\omega) = \sum_{i=1}^{r} \int_{B_i} X^*(\omega);$$

thus it suffices to prove that

$$(*) \qquad \int_{B_i} X^*(\omega) = \text{ index of } X \text{ at } p_i.$$

It will be convenient to drop the subscript i from now on.

We can assume that TM is trivial over B, so that $\pi^{-1}(B)$ can be identified with $B \times M_p$. Let j_p and π_2 have the same meaning as in the proof of Theorem 26. Also choose a norm $\| \;\|$ on M_p. We can assume that under the identification of $\pi^{-1}(B)$ with $B \times M_p$, the support of $\omega|\pi^{-1}(B)$ is contained in $\{(q, v) : q \in A, \|v\| \le 1\}$. Recall from the proof of Theorem 26 that

$$\pi_2{}^* j_p{}^* \omega - \omega = d\lambda \qquad\qquad \text{support } \lambda \subset \{(q, v) : \|v\| \le 1\}.$$

Since we can assume that $X(q) \notin$ support λ for $q \in \partial B$, we have

$$
\begin{aligned}
(1) \qquad \int_B X^*(\omega) &= \int_B X^*\pi_2{}^*(j_p{}^*\omega) - \int_B X^*(d\lambda) \\
&= \int_B X^*\pi_2{}^*(j_p{}^*\omega) - \int_{\partial B} X^*(\lambda) \qquad \text{by Stokes' Theorem} \\
&= \int_B X^*\pi_2{}^*(j_p{}^*\omega).
\end{aligned}
$$

On the manifold M_p we have

$$j_p{}^*\omega = d\rho \qquad \begin{array}{l}\rho \text{ an } (n-1)\text{-form on } M_p \\ \text{(with non-compact support).}\end{array}$$

If $D \subset M_p$ is the unit disc (with respect to the norm $\| \;\|$) and S^{n-1} denotes $\partial D \subset M_p$, then

$$
\begin{aligned}
(2) \qquad \int_{S^{n-1}} \rho = \int_{\partial D} \rho &= \int_D d\rho \\
&= \int_D j_p{}^*\omega \\
&= 1. \qquad \begin{array}{l}\text{by Theorem 26, and the fact} \\ \text{that support } j_p{}^*\omega \subset D.\end{array}
\end{aligned}
$$

Now, for $q \in B - \{p\}$, we can define

$$\bar{X}(q) = X(q)/|X(q)|,$$

and $\bar{X}\colon \partial B \to TM$ is smoothly homotopic to $X\colon \partial B \to TM$. So

(3)
$$\begin{aligned}\int_B X^*\pi_2{}^*(j_p{}^*\omega) &= \int_B X^*\pi_2{}^*\,d\rho \\ &= \int_{\partial B} X^*\pi_2{}^*\rho \qquad \text{by Stokes' Theorem} \\ &= \int_{\partial B} \bar{X}^*\pi_2{}^*\rho \\ &= \int_{\partial B} (\pi_2 \circ \bar{X})^*\rho.\end{aligned}$$

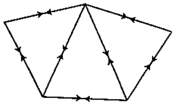
From the definition of the index of a vector field, together with equation (2), it follows that

(4)
$$\int_{\partial B} (\pi_2 \circ \bar{X}^*)\rho = \text{ index of } X \text{ at } p.$$

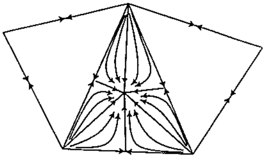
Equations (1), (3), (4) together imply (∗). ❖

29. COROLLARY. If X and Y are two vector fields with only finitely many zeros on a compact orientable manifold, then the sum of the indices of X equals the sum of the indices of Y.

At the moment, we do not even know that there is a vector field on M with finitely many zeros, nor do we know what this constant sum of the indices is (although our terminology certainly suggests a good guess). To resolve these questions, we consider once again a triangulation of M. We can then find a vector field X with just one zero in each k-simplex of the triangulation. We begin by drawing the integral curves of X along the 1-simplexes, with a zero at each 0-simplex and at one point in each 1-simplex. We then extend this picture



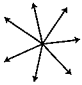
to include the integral curves of X on the 2-simplexes, producing a zero at one



point in each of them. We then continue similarly until the n-simplexes are filled.

30. THEOREM (POINCARÉ-HOPF). The sum of the indices of this vector field (and hence of any vector field) on M is the Euler characteristic $\chi(M)$. Thus, for $\xi = \pi : TM \to M$ we have $\chi(\xi) = \chi(M) \cdot \mu$.

PROOF. At each 0-simplex of the triangulation, the vector field looks like



with index 1.

Now consider the vector field in a neighborhood of the place where it is zero on a 1-simplex. The vector field looks like a vector field on $\mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1}$ which points directly inwards on $\mathbb{R}^1 \times \{0\}$ and directly outwards on $\{0\} \times \mathbb{R}^{n-1}$.

(a) $n = 2$ (b) $n = 3$