5. THEOREM. Let X be a C^∞ vector field on M, and let $p \in M$. Then there is an open set V containing p and an $\varepsilon > 0$, such that there is a unique collection of diffeomorphisms $\phi_t \colon V \to \phi_t(V) \subset M$ for $|t| < \varepsilon$ with the following properties:

(1) $\phi\colon (-\varepsilon, \varepsilon) \times V \to M$, defined by $\phi(t, p) = \phi_t(p)$, is C^∞.

(2) If $|s|, |t|, |s + t| < \varepsilon$, and $q, \phi_t(q) \in V$, then

$$\phi_{s+t}(q) = \phi_s \circ \phi_t(q).$$

(3) If $q \in V$, then X_q is the tangent vector at $t = 0$ of the curve $t \mapsto \phi_t(q)$.

The examples given previously show that we cannot expect ϕ_t to be defined for all t, or on all of M. In one case however, this can be attained. The support of a vector field X is just the closure of $\{p \in M : X_p \neq 0\}$.

6. THEOREM. If X has compact support (in particular, if M is compact), then there are diffeomorphisms $\phi_t \colon M \to M$ for all $t \in \mathbb{R}$ with properties (1), (2), (3).

PROOF. Cover support X by a finite number of open sets $V_1, \ldots, V_n$ given by Theorem 5 with corresponding $\varepsilon_1, \ldots, \varepsilon_n$ and diffeomorphisms ϕ_t^i. Let $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_n)$. Notice that by uniqueness, $\phi_t^i(q) = \phi_t^j(q)$ for $q \in V_i \cap V_j$. So we can define

$$\phi_t(q) = \begin{cases} \phi_t^i(q) & \text{if } q \in V_i \\ q & \text{if } q \notin \text{support } X. \end{cases}$$

Clearly $\phi\colon (-\varepsilon, \varepsilon) \times M \to M$ is C^∞, and $\phi_{t+s} = \phi_t \circ \phi_s$ if $|t|, |s|, |t + s| < \varepsilon$, and each ϕ_t is a diffeomorphism.

To define ϕ_t for $|t| \geq \varepsilon$, write

$$t = k(\varepsilon/2) + r \qquad \text{with } k \text{ an integer, and } |r| < \varepsilon/2.$$

Let

$$\phi_t = \begin{cases} \phi_{\varepsilon/2} \circ \cdots \circ \phi_{\varepsilon/2} \circ \phi_r & [\phi_{\varepsilon/2} \text{ iterated } k \text{ times}] \quad \text{for } k \geq 0 \\ \phi_{-\varepsilon/2} \circ \cdots \circ \phi_{-\varepsilon/2} \circ \phi_r & [\phi_{-\varepsilon/2} \text{ iterated } -k \text{ times}] \quad \text{for } k < 0. \end{cases}$$

It is easy to check that this is the desired $\{\phi_t\}$. ❖

The unique collection $\{\phi_t\}$ given by Theorem 6, or more precisely, the map $t \mapsto \phi_t$ from $\mathbb{R}$ to the group of all diffeomorphisms of M, is called a 1-parameter group of diffeomorphisms, and is said to be **generated** by X. In the local case of Theorem 5, we obtain a "local 1-parameter group of local diffeomorphisms". The vector field X is sometimes called the "infinitesimal generator" of $\{\phi_t\}$ (vector fields used to be called "infinitesimal transformations").

Condition (3) in Theorem 5 can be rephrased in terms of the action of X_q on a C^∞ function $f\colon M \to \mathbb{R}$. Recall that

$$\frac{dc}{dt}(f) = \frac{df(c(t))}{dt} = (f \circ c)'(t).$$

Thus, to say that X_q is the tangent vector at $t = 0$ of the curve $t \mapsto \phi_t(q)$ amounts to saying that

$$(Xf)(q) = X_q f = \lim_{h \to 0} \frac{f(\phi_h(q)) - f(q)}{h}.$$

This equation will be used very frequently. The first use is to derive a corollary of Theorem 5 which allows us to simplify many calculations involving vector fields, and which also has important theoretical uses.

7. THEOREM. Let X be a C^∞ vector field on M with $X(p) \neq 0$. Then there is a coordinate system (x, U) around p such that

$$X = \frac{\partial}{\partial x^1} \quad \text{on} \quad U.$$

PROOF. It is easy to see that we can assume $M = \mathbb{R}^n$ (with the standard coordinate system $t^1, \ldots, t^n$, say), and $p = 0 \in \mathbb{R}^n$. Moreover, we can assume that $X(0) = \partial/\partial t^1|_0$. The idea of the proof is that in a neighborhood of 0 there is a unique integral curve through each point $(0, a^2, \ldots, a^n)$; if q lies on the integral